Advanced Macroeconomics: Mid-term Exam

Growth Theory: Solow–Swan, Diamond OLG, RCK, and Romer Models
Time Allowed: 120 minutes — Total: 100 marks

Instructions. Answer all three questions. Show all mathematical derivations clearly. Answers that state final results without derivation will receive limited credit. You may use a calculator. Unless otherwise stated, assume all variables are positive and all parameters lie in economically meaningful ranges.

Question	Core Models	Marks
Question 1	Solow–Swan, convergence, growth accounting	35
Question 2	Diamond OLG, ageing, saving, dynamic efficiency	35
Question 3	RCK and Romer, optimal saving, R&D-based growth	30
Total		**100**

Question 1: Cross-Country Income Gaps, Conditional Convergence, and Kaldor Facts [Total: 35 marks]

A central stylised fact in growth economics is that income per capita differs enormously across countries. Some economies experienced rapid catch-up growth, while others remained far behind the world technology frontier. At the same time, advanced economies display approximately stable factor shares, no obvious long-run trend in the real interest rate, and sustained growth in output per worker.

Consider a discrete-time Solow–Swan economy with Cobb–Douglas production:

$$Y_t = K_t^{\alpha}(A_tL_t)^{1-\alpha}, \qquad 0<\alpha<1,$$

where

$$A_{t+1} = (1+g)A_t, \qquad L_{t+1} = (1+n)L_t.$$

Capital evolves according to

$$K_{t+1} = (1-\delta)K_t + sY_t, \qquad 0<s<1, \qquad 0<\delta<1.$$

Define

$$k_t \equiv \frac{K_t}{A_tL_t}, \qquad y_t \equiv \frac{Y_t}{A_tL_t}, \qquad c_t \equiv \frac{C_t}{A_tL_t},$$

and let

$$D \equiv (1+n)(1+g), \qquad b \equiv D-(1-\delta) = n+g+ng+\delta.$$

- **(a)** Derive the intensive-form production function and the exact law of motion:

  $$k_{t+1} = \mathcal{G}(k_t).$$

  Then derive $k_{t+1}-k_t$ and interpret the economic meaning of bk_t. **[7 marks]**
- **(b)** Solve for the positive steady-state values k^*, y^*, and c^*. Then derive the comparative statics of k^* and y^* with respect to s, n, g, and δ. **[8 marks]**
- **(c)** Suppose two countries share the same technology growth rate g and depreciation rate δ, but differ in saving and population growth:

  $$(s_H, n_H) = (0.30, 0.005), \qquad (s_L, n_L) = (0.10, 0.025),$$

  with

  $$\alpha = \frac{1}{3}, \qquad g = 0.02, \qquad \delta = 0.05.$$

  Using the exact formula for b, compute the long-run ratio

  $$\frac{(Y/L)_H^*}{(Y/L)_L^*}$$

  at a common technology level A_t. Interpret the result as a Solow-style explanation of cross-country income gaps. **[7 marks]**
- **(d)** Prove conditional convergence around the steady state. Derive the local convergence coefficient

  $$\lambda = \mathcal{G}'(k^*).$$

  Then explain why, conditional on the same structural parameters, a poorer economy grows faster than a richer economy. **[7 marks]**
- **(e)** Use the production function to derive the growth-accounting decomposition for output per worker:

  $$\Delta\ln(Y_t/L_t) = \alpha\Delta\ln(K_t/L_t) + (1-\alpha)\Delta\ln A_t.$$

  If an advanced economy has

  $$\Delta\ln(Y_t/L_t) = 0.02, \qquad \Delta\ln(K_t/L_t) = 0.02, \qquad \alpha = \frac{1}{3},$$

  infer the implied growth rate of labour-augmenting technology. Explain how this relates to the Kaldor facts. **[6 marks]**

Question 2: Ageing, Endogenous Saving, and Dynamic Efficiency in the Diamond OLG Model [Total: 35 marks]

Many advanced economies face population ageing: fertility rates have fallen, dependency ratios have risen, and governments are under pressure to reform pension systems. This question uses the Diamond OLG model to study how ageing and pay-as-you-go pensions affect saving, capital accumulation, and dynamic efficiency.

Individuals live for two periods. Preferences are

$$U(c_{1t}, c_{2,t+1}) = u(c_{1t}) + \beta u(c_{2,t+1}), \qquad 0 < \beta < 1,$$

with

$$u(c) = \frac{c^{1-\theta} - 1}{1-\theta}, \qquad \theta > 0.$$

Without pensions,

$$c_{1t} + s_t = A_t w_t, \qquad c_{2,t+1} = (1 + r_{t+1})s_t.$$

Firms operate

$$Y_t = A_t L_t f(k_t), \qquad f(k) = k^\alpha, \qquad 0 < \alpha < 1,$$

where

$$k_t = \frac{K_t}{A_t L_t}, \qquad A_{t+1} = (1+g)A_t, \qquad L_{t+1} = (1+n)L_t.$$

There is no capital depreciation. Let

$$D \equiv (1+n)(1+g).$$

(a) Solve the young household's saving problem. Derive the Euler equation and show that

$$s_t = \sigma(r_{t+1})A_t w_t,$$

where

$$\sigma(r) = \frac{\beta^{1/\theta}}{\beta^{1/\theta} + (1+r)^{1-1/\theta}}.$$

Then derive $\sigma'(r)$ and discuss the cases $\theta < 1$, $\theta = 1$, and $\theta > 1$. **[10 marks]**

(b) Derive the firm's first-order conditions:

$$r_t = f'(k_t), \qquad w_t = f(k_t) - f'(k_t)k_t.$$

Using market clearing,

$$L_t^D = L_t, \qquad K_{t+1} = L_t s_t,$$

derive the implicit transition equation:

$$k_{t+1} = \frac{1}{D}\sigma\big(f'(k_{t+1})\big)\left[f(k_t) - f'(k_t)k_t\right].$$

[7 marks]

(c) Now impose log utility, $\theta = 1$. Derive

$$k_{t+1} = Bk_t^\alpha, \qquad B \equiv \frac{\beta}{1+\beta}\frac{1-\alpha}{D}.$$

Solve for k^*. Then derive the Golden Rule capital stock k_{GR} and the dynamic-efficiency condition. **[8 marks]**

(d) Consider an ageing shock that lowers population growth from n_0 to $n_1 < n_0$, holding all other parameters fixed. Derive analytically how this shock affects D, B, k^*, k_{GR}, and $r^* = f'(k^*)$. Does population ageing necessarily generate dynamic inefficiency? Explain carefully. **[5 marks]**

(e) Suppose the government introduces a pay-as-you-go pension. Each young individual pays a tax τA_t when young. When old, the individual receives

$$T_{t+1} = (1+n)\tau A_{t+1}.$$

The budget constraints become

$$c_{1t} + s_t = A_t w_t - \tau A_t,$$

$$c_{2,t+1} = (1 + r_{t+1})s_t + (1+n)\tau A_{t+1}.$$

Assume log utility. Derive the optimal saving function s_t as a function of w_t, r_{t+1}, τ, D, and A_t. Show mathematically why a positive PAYG tax reduces private saving. **[5 marks]**

Question 3: Optimal Saving, Innovation Policy, and Endogenous Growth [Total: 30 marks]

Suppose a government is concerned that capital accumulation alone cannot explain sustained growth in living standards. It considers an innovation strategy that reallocates skilled workers from final-goods production into R&D. This question compares the RCK model, where technological growth is exogenous, with the Romer model, where technological progress is produced by researchers.

Part I: RCK Economy

There is no population growth and no depreciation. Technology grows at

$$A_{t+1} = (1+g)A_t.$$

Households have preferences

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \qquad u(c) = \frac{c^{1-\theta} - 1}{1-\theta}.$$

Production is

$$Y_t = A_t L f(k_t), \qquad f(k) = k^{\alpha}, \qquad k_t = \frac{K_t}{A_t L}, \qquad \widetilde{c}_t = \frac{c_t}{A_t}.$$

(a) Derive the RCK transition equations:

$$\widetilde{c}_t = f(k_t) + k_t - (1+g)k_{t+1},$$

and

$$\left(\frac{\widetilde{c}_{t+1}}{\widetilde{c}_t}\right)^{\theta} = \frac{\beta[1 + f'(k_{t+1})]}{(1+g)^{\theta}}.$$

Then solve for k^* under Cobb–Douglas production. **[8 marks]**

(b) Derive the Golden Rule capital stock:

$$k_{\text{GR}} = \left(\frac{\alpha}{g}\right)^{1/(1-\alpha)}.$$

Under

$$\beta(1+g)^{1-\theta} < 1,$$

prove that

$$k^* < k_{\text{GR}}.$$

Give the economic intuition. **[6 marks]**

(c) Rewrite the RCK model as a two-dimensional first-order system in $(k_t, \widetilde{c}_t)$. Derive the $\Delta k = 0$ and $\Delta \widetilde{c} = 0$ loci. Explain why the steady state is saddle-path stable and why the transversality condition matters. **[6 marks]**

Part II: Romer Economy

Now suppose technology is endogenous:

$$Y_t = A_t K_t^{\alpha} L_{y,t}^{1-\alpha}, \qquad A_{t+1} - A_t = \bar{z} A_t L_{a,t}.$$

Total labour is fixed:

$$\bar{L}_t = \bar{L}, \qquad L_{a,t} = \bar{\ell}\bar{L}, \qquad L_{y,t} = (1-\bar{\ell})\bar{L}.$$

Capital evolves as

$$K_{t+1} - K_t = sY_t - \delta K_t.$$

(d) Derive the growth rate of ideas,

$$g_A = \frac{A_{t+1} - A_t}{A_t}.$$

Then prove that

$$g_Y = g_K = g_y = \frac{\bar{z}\bar{\ell}\bar{L}}{1-\alpha}.$$

Explain why the Romer model can generate sustained per-capita growth even when population is constant. **[6 marks]**

(e) Suppose

$$\alpha = \frac{1}{3}, \qquad \bar{z} = 0.0005, \qquad \bar{L} = 1000.$$

The government raises the research-labour share from

$$\bar{\ell}_0 = 0.02$$

to

$$\bar{\ell}_1 = 0.05.$$

Compute the old and new values of g_A and g_y. Then compute the immediate effect on output per person, holding A_t and $K_t/\bar{L}$ fixed:

$$\frac{y_t(\bar{\ell}_1)}{y_t(\bar{\ell}_0)}.$$

Interpret the short-run cost and long-run benefit of the innovation policy. **[4 marks]**

Detailed Solutions

Solution to Question 1 [35 marks]

Part (a) [7 marks]

The production function is

$$Y_t = K_t^\alpha (A_t L_t)^{1-\alpha}.$$

Divide by $A_t L_t$:

$$y_t = \frac{Y_t}{A_t L_t} = \frac{K_t^\alpha (A_t L_t)^{1-\alpha}}{A_t L_t} = \left(\frac{K_t}{A_t L_t}\right)^\alpha.$$

Therefore,

$$\boxed{y_t = k_t^\alpha.}$$

Goods-market clearing implies investment equals saving:

$$I_t = sY_t.$$

Capital accumulation is

$$K_{t+1} = (1-\delta)K_t + sY_t.$$

Divide by $A_t L_t$:

$$\frac{K_{t+1}}{A_t L_t} = (1-\delta)k_t + sk_t^\alpha.$$

Since

$$A_{t+1}L_{t+1} = (1+g)(1+n)A_t L_t = DA_t L_t,$$

we have

$$\frac{K_{t+1}}{A_t L_t} = Dk_{t+1}.$$

Hence,

$$Dk_{t+1} = (1-\delta)k_t + sk_t^\alpha.$$

Thus,

$$\boxed{k_{t+1} = \mathcal{G}(k_t) = \frac{(1-\delta)k_t + sk_t^\alpha}{D}.}$$

Subtracting k_t from both sides:

$$k_{t+1} - k_t = \frac{(1-\delta)k_t + sk_t^\alpha - Dk_t}{D}.$$

Since

$$b = D - (1-\delta),$$

we obtain

$$\boxed{k_{t+1} - k_t = \frac{sk_t^\alpha - bk_t}{D}.}$$

The term sk_t^α is actual investment per unit of effective labour. The term bk_t is break-even investment: the investment required to keep k_t constant despite depreciation, population growth, technological progress, and the interaction term ng.

Marking guide: intensive-form production, 2; transition equation, 3; $k_{t+1} - k_t$ and interpretation, 2.

Part (b) [8 marks]

At a positive steady state,

$$k_{t+1} = k_t = k^*.$$

Thus,

$$k^* = \frac{(1-\delta)k^* + s(k^*)^\alpha}{D}.$$

Multiplying by D and rearranging:

$$[D - (1-\delta)]\,k^* = s(k^*)^\alpha.$$

Thus,

$$bk^* = s(k^*)^\alpha.$$

For $k^* > 0$:

$$b(k^*)^{1-\alpha} = s.$$

Hence,

$$\boxed{k^* = \left(\frac{s}{b}\right)^{1/(1-\alpha)}.}$$

Then

$$\boxed{y^* = (k^*)^\alpha = \left(\frac{s}{b}\right)^{\alpha/(1-\alpha)}.}$$

Since

$$c_t = (1-s)y_t,$$

$$\boxed{c^* = (1-s)\left(\frac{s}{b}\right)^{\alpha/(1-\alpha)}.}$$

For comparative statics, take logs:

$$\ln k^* = \frac{1}{1-\alpha}(\ln s - \ln b).$$

Thus,

$$\boxed{\frac{\partial \ln k^*}{\partial \ln s} = \frac{1}{1-\alpha} > 0.}$$

Since

$$b = (1+n)(1+g) - (1-\delta),$$

we have

$$\frac{\partial b}{\partial n} = 1+g, \qquad \frac{\partial b}{\partial g} = 1+n, \qquad \frac{\partial b}{\partial \delta} = 1.$$

Therefore,

$$\boxed{\frac{\partial \ln k^*}{\partial n} = -\frac{1+g}{(1-\alpha)b} < 0,}$$

$$\boxed{\frac{\partial \ln k^*}{\partial g} = -\frac{1+n}{(1-\alpha)b} < 0,}$$

and

$$\boxed{\frac{\partial \ln k^*}{\partial \delta} = -\frac{1}{(1-\alpha)b} < 0.}$$

Since

$$\ln y^* = \alpha \ln k^*,$$

we have

$$\boxed{\frac{\partial \ln y^*}{\partial x} = \alpha \frac{\partial \ln k^*}{\partial x}, \qquad x \in \{s, n, g, \delta\}.}$$

Marking guide: steady-state equation, 2; k^*, y^*, c^*, 3; comparative statics for k^*, 2; comparative statics for y^*, 1.

Part (c) [7 marks]

For country H,

$$b_H = (1+n_H)(1+g) - (1-\delta).$$

Using

$$n_H = 0.005, \qquad g = 0.02, \qquad \delta = 0.05,$$

we get

$$b_H = (1.005)(1.02) - 0.95 = 1.0251 - 0.95 = 0.0751.$$

For country L,

$$b_L = (1+n_L)(1+g) - (1-\delta).$$

Using

$$n_L = 0.025,$$

$$b_L = (1.025)(1.02) - 0.95 = 1.0455 - 0.95 = 0.0955.$$

At a common technology level A_t,

$$\frac{Y_t}{L_t} = A_t y^* = A_t \left(\frac{s}{b}\right)^{\alpha/(1-\alpha)}.$$

Therefore,

$$\frac{(Y/L)^*_H}{(Y/L)^*_L} = \left[\frac{s_H/b_H}{s_L/b_L}\right]^{\alpha/(1-\alpha)}.$$

With

$$\alpha = \frac{1}{3}, \qquad \frac{\alpha}{1-\alpha} = \frac{1}{2},$$

we get

$$\frac{(Y/L)^*_H}{(Y/L)^*_L} = \left[\frac{0.30/0.0751}{0.10/0.0955}\right]^{1/2}.$$

Compute:

$$\frac{0.30}{0.0751} \approx 3.9947, \qquad \frac{0.10}{0.0955} \approx 1.0471.$$

Thus,

$$\frac{(Y/L)^*_H}{(Y/L)^*_L} = \sqrt{\frac{3.9947}{1.0471}} \approx \sqrt{3.815} \approx 1.953.$$

Therefore,

$$\boxed{\frac{(Y/L)^*_H}{(Y/L)^*_L} \approx 1.95.}$$

The high-saving, low-population-growth economy has almost twice the long-run output per worker of the low-saving, high-population-growth economy, even when both countries share the same technology level. The Solow model therefore explains part of cross-country income differences through differences in investment effort and capital dilution.

Marking guide: correct b_H and b_L, 2; ratio formula, 3; numerical ratio, 1; interpretation, 1.

Part (d) **[7 marks]**

The transition function is

$$\mathcal{G}(k) = \frac{(1-\delta)k + sk^\alpha}{D}.$$

Differentiate:

$$\mathcal{G}'(k) = \frac{(1-\delta) + s\alpha k^{\alpha-1}}{D}.$$

At the steady state,

$$s(k^*)^{\alpha-1} = b.$$

Thus,

$$\boxed{\lambda = \mathcal{G}'(k^*) = \frac{(1-\delta) + \alpha b}{D}.}$$

To prove convergence, use

$$k_{t+1} - k_t = \frac{k_t(sk_t^{\alpha-1} - b)}{D}.$$

Since $\alpha - 1 < 0$, $k^{\alpha-1}$ is strictly decreasing. At k^*,

$$s(k^*)^{\alpha-1} = b.$$

If $k_t < k^*$, then

$$k_t^{\alpha-1} > (k^*)^{\alpha-1},$$

so

$$sk_t^{\alpha-1} > b,$$

and therefore

$$k_{t+1} - k_t > 0.$$

If $k_t > k^*$, then

$$sk_t^{\alpha-1} < b,$$

so

$$k_{t+1} - k_t < 0.$$

Thus, the economy converges to k^*.

Conditional convergence means that, among economies with the same structural parameters $(s, n, g, \delta, \alpha)$, the one further below its steady state has a larger gap between actual investment and break-even investment. Hence it experiences faster capital deepening and faster output-per-worker growth.

Marking guide: derivative and λ, 3; convergence proof, 3; conditional-convergence interpretation, 1.

Part (e) **[6 marks]**

Output per worker is

$$\frac{Y_t}{L_t} = \frac{K_t^{\alpha}(A_t L_t)^{1-\alpha}}{L_t} = \left(\frac{K_t}{L_t}\right)^{\alpha} A_t^{1-\alpha}.$$

Taking log differences:

$$\Delta \ln(Y_t/L_t) = \alpha \Delta \ln(K_t/L_t) + (1-\alpha)\Delta \ln A_t.$$

Substitute

$$\Delta \ln(Y_t/L_t) = 0.02, \qquad \Delta \ln(K_t/L_t) = 0.02, \qquad \alpha = \frac{1}{3}.$$

Then

$$0.02 = \frac{1}{3}(0.02) + \frac{2}{3}\Delta \ln A_t.$$

Therefore,

$$0.02 - \frac{0.02}{3} = \frac{2}{3}\Delta \ln A_t.$$

The left-hand side is

$$0.02 - 0.006667 = 0.013333.$$

Thus,

$$\Delta \ln A_t = \frac{0.013333}{2/3} = 0.02.$$

Hence,

$$\boxed{\Delta \ln A_t = 0.02.}$$

This is consistent with the Kaldor facts: output per worker and capital per worker grow at similar long-run rates, while the capital-output ratio is approximately stable. In the Solow model, this requires sustained labour-augmenting technological progress.

Marking guide: growth-accounting derivation, 3; numerical technology growth, 2; Kaldor-facts interpretation, 1.

Solution to Question 2 [35 marks]

Part (a) [10 marks]

The young household chooses s_t to maximise

$$u(A_t w_t - s_t) + \beta u((1+r_{t+1})s_t).$$

The first-order condition is

$$-u'(c_{1t}) + \beta u'(c_{2,t+1})(1+r_{t+1}) = 0.$$

Thus,

$$\boxed{u'(c_{1t}) = \beta(1+r_{t+1})u'(c_{2,t+1}).}$$

For CRRA utility,

$$u'(c) = c^{-\theta}.$$

Therefore,

$$(A_t w_t - s_t)^{-\theta} = \beta(1+r_{t+1})\left[(1+r_{t+1})s_t\right]^{-\theta}.$$

This implies

$$(A_t w_t - s_t)^{-\theta} = \beta(1+r_{t+1})^{1-\theta} s_t^{-\theta}.$$

Taking both sides to the power $-1/\theta$:

$$A_t w_t - s_t = \beta^{-1/\theta}(1+r_{t+1})^{1-1/\theta} s_t.$$

Thus,

$$A_t w_t = s_t\left[1 + \beta^{-1/\theta}(1+r_{t+1})^{1-1/\theta}\right].$$

Hence,

$$s_t = \frac{A_t w_t}{1 + \beta^{-1/\theta}(1+r_{t+1})^{1-1/\theta}}.$$

Multiplying numerator and denominator by $\beta^{1/\theta}$ gives

$$\boxed{s_t = \frac{\beta^{1/\theta}}{\beta^{1/\theta} + (1+r_{t+1})^{1-1/\theta}} A_t w_t.}$$

Therefore,

$$\boxed{s_t = \sigma(r_{t+1}) A_t w_t.}$$

Now define

$$a = 1 - \frac{1}{\theta}.$$

Then

$$\sigma(r) = \frac{\beta^{1/\theta}}{\beta^{1/\theta} + (1+r)^a}.$$

Differentiate:

$$\sigma'(r) = -\frac{\beta^{1/\theta} a (1+r)^{a-1}}{\left[\beta^{1/\theta} + (1+r)^a\right]^2}.$$

Since

$$-a = \frac{1}{\theta} - 1, \qquad a - 1 = -\frac{1}{\theta},$$

we get

$$\boxed{\sigma'(r) = \left(\frac{1}{\theta} - 1\right) \frac{\beta^{1/\theta}(1+r)^{-1/\theta}}{\left[\beta^{1/\theta} + (1+r)^{1-1/\theta}\right]^2}.}$$

Therefore,

$$\boxed{\sigma'(r) \begin{cases} > 0, & \theta < 1, \\ = 0, & \theta = 1, \\ < 0, & \theta > 1. \end{cases}}$$

If $\theta < 1$, the intertemporal elasticity of substitution is high, and the substitution effect dominates: a higher return raises saving. If $\theta > 1$, the income effect dominates for a saver: a higher return allows the same old-age consumption with less saving. If $\theta = 1$, the two effects exactly offset.

Marking guide: Euler equation, 2; saving function, 5; derivative, 2; interpretation, 1.

Part (b) [7 marks]

The firm solves

$$\max_{K_t,L_t} A_tL_tf(k_t) - r_tK_t - w_tA_tL_t, \qquad k_t = \frac{K_t}{A_tL_t}.$$

The first-order condition with respect to capital is

$$\boxed{r_t = f'(k_t).}$$

The wage per unit of effective labour is

$$\boxed{w_t = f(k_t) - f'(k_t)k_t.}$$

Capital-market clearing gives

$$K_{t+1} = L_ts_t.$$

Using

$$s_t = \sigma(r_{t+1})A_tw_t,$$

we get

$$K_{t+1} = L_t\sigma(r_{t+1})A_tw_t.$$

Divide by

$$A_{t+1}L_{t+1} = DA_tL_t.$$

Then

$$k_{t+1} = \frac{1}{D}\sigma(r_{t+1})w_t.$$

Using firm optimality,

$$r_{t+1} = f'(k_{t+1}), \qquad w_t = f(k_t) - f'(k_t)k_t.$$

Therefore,

$$\boxed{k_{t+1} = \frac{1}{D}\sigma(f'(k_{t+1}))\left[f(k_t) - f'(k_t)k_t\right].}$$

Marking guide: firm FOCs, 3; market-clearing derivation, 3; final implicit transition equation, 1.

Part (c) [8 marks]

When $\theta = 1$,

$$\sigma(r) = \frac{\beta}{1+\beta}.$$

With

$$f(k) = k^\alpha,$$

we have

$$f'(k) = \alpha k^{\alpha-1}.$$

The wage is

$$w_t = f(k_t) - f'(k_t)k_t = k_t^\alpha - \alpha k_t^\alpha = (1-\alpha)k_t^\alpha.$$

Therefore,

$$k_{t+1} = \frac{1}{D}\frac{\beta}{1+\beta}(1-\alpha)k_t^\alpha.$$

Define

$$\boxed{B \equiv \frac{\beta}{1+\beta}\frac{1-\alpha}{D}.}$$

Then

$$\boxed{k_{t+1} = Bk_t^\alpha.}$$

A positive steady state satisfies

$$k^* = B(k^*)^\alpha.$$

Thus,

$$(k^*)^{1-\alpha} = B,$$

so

$$\boxed{k^* = B^{1/(1-\alpha)}.}$$

In the Diamond model with no depreciation, steady-state consumption per effective worker is

$$c(k) = f(k) - (D-1)k.$$

The Golden Rule solves

$$\max_k \left[f(k) - (D-1)k\right].$$

The first-order condition is

$$f'(k_{\text{GR}}) = D - 1.$$

For Cobb–Douglas production:

$$\alpha k_{\text{GR}}^{\alpha-1} = D - 1.$$

Thus,

$$\boxed{k_{\text{GR}} = \left(\frac{\alpha}{D-1}\right)^{1/(1-\alpha)}.}$$

The economy is dynamically efficient if

$$\boxed{k^* \le k_{\text{GR}}.}$$

Equivalently,

$$\boxed{f'(k^*) \ge D - 1.}$$

Since $r^* = f'(k^*)$, this is

$$\boxed{r^* \ge D - 1.}$$

Dynamic inefficiency occurs when

$$\boxed{k^* > k_{\text{GR}} \quad \Longleftrightarrow \quad r^* < D - 1.}$$

Marking guide: log-utility transition, 3; steady state, 2; Golden Rule, 2; efficiency condition, 1.

Part (d) [5 marks]

An ageing shock lowers n, so

$$D = (1+n)(1+g)$$

falls. Since

$$B = \frac{\beta}{1+\beta}\frac{1-\alpha}{D},$$

a fall in D increases B:

$$\frac{\partial B}{\partial D} < 0.$$

Since

$$k^* = B^{1/(1-\alpha)},$$

we have

$$\boxed{D\downarrow \quad \Rightarrow \quad B\uparrow \quad \Rightarrow \quad k^*\uparrow.}$$

The Golden Rule is

$$k_{\text{GR}} = \left(\frac{\alpha}{D-1}\right)^{1/(1-\alpha)}.$$

When D falls, $D-1$ falls, so

$$\boxed{D\downarrow \quad \Rightarrow \quad k_{\text{GR}}\uparrow.}$$

The steady-state interest rate is

$$r^* = f'(k^*) = \alpha(k^*)^{\alpha-1}.$$

Since k^* rises and $f'(k)$ is decreasing,

$$\boxed{k^*\uparrow \quad \Rightarrow \quad r^*\downarrow.}$$

However, ageing does not necessarily generate dynamic inefficiency. Dynamic inefficiency requires

$$r^* < D - 1.$$

Ageing lowers r^*, but it also lowers the benchmark growth term $D-1$. In the log–Cobb-Douglas case, let

$$q = \frac{\beta}{1+\beta}.$$

Then

$$k^* = \left[\frac{q(1-\alpha)}{D}\right]^{1/(1-\alpha)}.$$

Overaccumulation requires

$$k^* > k_{\mathrm{GR}}.$$

This is equivalent to

$$\frac{q(1-\alpha)}{D} > \frac{\alpha}{D-1}.$$

Thus,

$$\boxed{q(1-\alpha)(D-1) > \alpha D.}$$

A lower D makes the left-hand side smaller relative to the right-hand side, so ageing alone does not mechanically imply dynamic inefficiency.

Marking guide: effects on $D, B, k^*, k_{\mathrm{GR}}, r^*$, 4; dynamic-inefficiency explanation, 1.

Part (e) [5 marks]

Under PAYG pensions and log utility, the household maximises

$$\ln c_{1t} + \beta \ln c_{2,t+1}$$

subject to

$$c_{1t} = A_t w_t - \tau A_t - s_t = A_t(w_t - \tau) - s_t,$$

and

$$c_{2,t+1} = (1 + r_{t+1})s_t + (1+n)\tau A_{t+1}.$$

Since

$$A_{t+1} = (1+g)A_t,$$

we have

$$(1+n)A_{t+1} = DA_t.$$

Therefore,

$$c_{2,t+1} = (1 + r_{t+1})s_t + D\tau A_t.$$

The Euler equation for log utility is

$$\frac{1}{c_{1t}} = \beta(1 + r_{t+1})\frac{1}{c_{2,t+1}}.$$

Hence,

$$c_{2,t+1} = \beta(1 + r_{t+1})c_{1t}.$$

Substitute the two budget constraints:

$$(1 + r_{t+1})s_t + D\tau A_t = \beta(1 + r_{t+1})\left[A_t(w_t - \tau) - s_t\right].$$

Expand:

$$(1 + r_{t+1})s_t + D\tau A_t = \beta(1 + r_{t+1})A_t(w_t - \tau) - \beta(1 + r_{t+1})s_t.$$

Collect terms in s_t:

$$(1+\beta)(1 + r_{t+1})s_t = \beta(1 + r_{t+1})A_t(w_t - \tau) - D\tau A_t.$$

Thus,

$$s_t = \frac{\beta}{1+\beta}A_t(w_t - \tau) - \frac{D\tau A_t}{(1+\beta)(1 + r_{t+1})}.$$

Therefore,

$$\boxed{s_t = A_t\left[\frac{\beta}{1+\beta}w_t - \frac{\tau}{1+\beta}\left(\beta + \frac{D}{1 + r_{t+1}}\right)\right].}$$

Without pensions,

$$s_t^0 = \frac{\beta}{1+\beta}A_t w_t.$$

With pensions,

$$s_t^{PAYG} = s_t^0 - A_t\frac{\tau}{1+\beta}\left(\beta + \frac{D}{1 + r_{t+1}}\right).$$

Since

$$A_t > 0, \qquad \tau > 0, \qquad \beta > 0, \qquad D > 0, \qquad 1 + r_{t+1} > 0,$$

the second term is strictly positive. Therefore,

$$\boxed{s_t^{PAYG} < s_t^0.}$$

PAYG pensions reduce private saving because they both lower disposable income when young and provide retirement income when old.

Marking guide: modified budgets, 1; log Euler equation, 1; saving function, 2; proof that PAYG reduces saving, 1.

Solution to Question 3 [30 marks]

Part (a) [8 marks]

The aggregate resource constraint is

$$Y_t = C_t + K_{t+1} - K_t,$$

because there is no depreciation. Divide by $A_t L$:

$$\frac{Y_t}{A_t L} = \frac{C_t}{A_t L} + \frac{K_{t+1}}{A_t L} - \frac{K_t}{A_t L}.$$

Since

$$\frac{Y_t}{A_t L} = f(k_t), \qquad \frac{C_t}{A_t L} = \widetilde{c}_t, \qquad \frac{K_t}{A_t L} = k_t,$$

and

$$\frac{K_{t+1}}{A_t L} = \frac{K_{t+1}}{A_{t+1} L}\frac{A_{t+1}}{A_t} = (1+g)k_{t+1},$$

we obtain

$$f(k_t) = \widetilde{c}_t + (1+g)k_{t+1} - k_t.$$

Thus,

$$\boxed{\widetilde{c}_t = f(k_t) + k_t - (1+g)k_{t+1}.}$$

The household Euler equation is

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = 1 + r_{t+1}.$$

Competitive firms imply

$$r_{t+1} = f'(k_{t+1}).$$

Since

$$c_t = A_t \widetilde{c}_t, \qquad c_{t+1} = A_{t+1}\widetilde{c}_{t+1},$$

and CRRA utility implies

$$u'(c) = c^{-\theta},$$

we have

$$\frac{(A_t \widetilde{c}_t)^{-\theta}}{\beta (A_{t+1}\widetilde{c}_{t+1})^{-\theta}} = 1 + f'(k_{t+1}).$$

This implies

$$\frac{1}{\beta}\left(\frac{A_{t+1}\widetilde{c}_{t+1}}{A_t \widetilde{c}_t}\right)^{\theta} = 1 + f'(k_{t+1}).$$

Since

$$A_{t+1} = (1+g)A_t,$$

we obtain

$$\boxed{\left(\frac{\widetilde{c}_{t+1}}{\widetilde{c}_t}\right)^{\theta} = \frac{\beta[1+f'(k_{t+1})]}{(1+g)^{\theta}}.}$$

At a steady state,

$$k_{t+1} = k_t = k^*, \qquad \widetilde{c}_{t+1} = \widetilde{c}_t = \widetilde{c}^*.$$

The Euler equation gives

$$1 = \frac{\beta[1+f'(k^*)]}{(1+g)^{\theta}}.$$

Thus,

$$f'(k^*) = \frac{(1+g)^{\theta}}{\beta} - 1.$$

For

$$f(k) = k^{\alpha},$$

we have

$$f'(k) = \alpha k^{\alpha-1}.$$

Hence,

$$\alpha (k^*)^{\alpha-1} = \frac{(1+g)^{\theta}}{\beta} - 1.$$

Therefore,

$$\boxed{k^* = \left[\frac{\alpha}{\frac{(1+g)^\theta}{\beta} - 1}\right]^{1/(1-\alpha)}.}$$

Marking guide: resource transition, 3; Euler transition, 3; steady-state k^*, 2.

Part (b) [6 marks]

Steady-state consumption per unit of effective labour is

$$\widetilde{c} = f(k) - gk.$$

The Golden Rule capital stock maximises

$$\max_k \left[f(k) - gk\right].$$

The first-order condition is

$$f'(k_{\mathrm{GR}}) = g.$$

For Cobb–Douglas production,

$$\alpha k_{\mathrm{GR}}^{\alpha-1} = g.$$

Thus,

$$k_{\mathrm{GR}}^{1-\alpha} = \frac{\alpha}{g}.$$

Therefore,

$$\boxed{k_{\mathrm{GR}} = \left(\frac{\alpha}{g}\right)^{1/(1-\alpha)}.}$$

The modified Golden Rule satisfies

$$f'(k^*) = \frac{(1+g)^\theta}{\beta} - 1.$$

The Golden Rule satisfies

$$f'(k_{\mathrm{GR}}) = g.$$

The restriction

$$\beta(1+g)^{1-\theta} < 1$$

implies

$$\beta < (1+g)^{\theta-1}.$$

Taking reciprocals:

$$\frac{1}{\beta} > (1+g)^{1-\theta}.$$

Multiplying by $(1+g)^\theta$:

$$\frac{(1+g)^\theta}{\beta} > 1 + g.$$

Thus,

$$\frac{(1+g)^\theta}{\beta} - 1 > g.$$

Therefore,

$$f'(k^*) > f'(k_{\mathrm{GR}}).$$

Since $f''(k) < 0$, the marginal product of capital is decreasing. Hence,

$$\boxed{k^* < k_{\mathrm{GR}}.}$$

The intuition is that the RCK steady state is the modified Golden Rule: households discount future utility, so the economy optimally chooses less capital than the level that would maximise steady-state consumption.

Marking guide: Golden Rule derivation, 2; proof that $k^* < k_{\mathrm{GR}}$, 3; intuition, 1.

Part (c) [6 marks]

The resource equation is

$$\widetilde{c}_t = f(k_t) + k_t - (1+g)k_{t+1}.$$

Solving for k_{t+1}:

$$\boxed{k_{t+1} = G(k_t, \widetilde{c}_t) = \frac{f(k_t) + k_t - \widetilde{c}_t}{1+g}.}$$

The Euler equation is

$$\left(\frac{\widetilde{c}_{t+1}}{\widetilde{c}_t}\right)^{\theta} = \frac{\beta[1 + f'(k_{t+1})]}{(1+g)^{\theta}}.$$

Thus,

$$\boxed{\widetilde{c}_{t+1} = H(k_t, \widetilde{c}_t) = \widetilde{c}_t \left[\frac{\beta[1 + f'(k_{t+1})]}{(1+g)^{\theta}}\right]^{1/\theta},}$$

where

$$k_{t+1} = G(k_t, \widetilde{c}_t).$$

The $\Delta k = 0$ locus is defined by

$$k_{t+1} = k_t.$$

Using the resource equation:

$$\widetilde{c}_t = f(k_t) - g k_t.$$

Thus,

$$\boxed{\Delta k = 0: \quad \widetilde{c} = f(k) - gk.}$$

The $\Delta \widetilde{c} = 0$ locus requires

$$\widetilde{c}_{t+1} = \widetilde{c}_t.$$

From the Euler equation:

$$\beta[1 + f'(k_{t+1})] = (1+g)^{\theta}.$$

Thus,

$$k_{t+1} = k^*.$$

Using the resource equation:

$$k^* = \frac{f(k_t) + k_t - \widetilde{c}_t}{1+g}.$$

Therefore,

$$\boxed{\Delta \widetilde{c} = 0: \quad \widetilde{c}_t = f(k_t) + k_t - (1+g)k^*.}$$

The steady state is saddle-path stable because capital k_t is predetermined, while consumption $\widetilde{c}_t$ is a forward-looking jump variable. Around the steady state, the linearised system has one stable eigenvalue and one unstable eigenvalue. Only one initial value of $\widetilde{c}_0$ places the economy on the stable saddle path.

The transversality condition rules out paths that violate household optimality, such as paths with explosive capital accumulation or paths in which households leave valuable capital unconsumed forever. It selects the economically meaningful saddle path.

Marking guide: first-order system, 2; $\Delta k = 0$ locus, 1; $\Delta \widetilde{c} = 0$ locus, 1; saddle-path and TVC explanation, 2.

Part (d) [6 marks]

The idea-production equation is

$$A_{t+1} - A_t = \bar{z} A_t L_{a,t}.$$

Divide by A_t:

$$\frac{A_{t+1} - A_t}{A_t} = \bar{z} L_{a,t}.$$

Since

$$L_{a,t} = \bar{\ell}\bar{L},$$

we get

$$\boxed{g_A = \bar{z}\bar{\ell}\bar{L}.}$$

The final-good production function is

$$Y_t = A_t K_t^{\alpha} L_{y,t}^{1-\alpha}.$$

Taking growth rates:

$$g_Y = g_A + \alpha g_K + (1-\alpha) g_{L_y}.$$

Since total labour is fixed and $\bar{\ell}$ is constant,

$$L_{y,t} = (1 - \bar{\ell})\bar{L}$$

is constant, so

$$g_{L_y} = 0.$$

Therefore,

$$g_Y = g_A + \alpha g_K.$$

Along a balanced-growth path, the output-capital ratio is constant:

$$\frac{Y_t}{K_t} = \text{constant}.$$

Thus,

$$g_Y = g_K.$$

Substitute $g_K = g_Y$ into the growth equation:

$$g_Y = g_A + \alpha g_Y.$$

Therefore,

$$(1-\alpha)g_Y = g_A.$$

Thus,

$$g_Y = \frac{g_A}{1-\alpha}.$$

Since population is constant,

$$g_y = g_Y.$$

Hence,

$$\boxed{g_Y = g_K = g_y = \frac{\bar{z}\bar{\ell}\bar{L}}{1-\alpha}.}$$

The Romer model generates sustained per-capita growth because ideas are nonrival and grow endogenously through R&D. Even when population is constant, a fixed number of researchers can keep expanding the stock of ideas, which raises productivity economy-wide.

Marking guide: g_A, 2; growth-rate algebra, 2; balanced-growth derivation, 1; interpretation, 1.

Part (e) [4 marks]

Given

$$\alpha = \frac{1}{3}, \qquad \bar{z} = 0.0005, \qquad \bar{L} = 1000.$$

Initially,

$$\bar{\ell}_0 = 0.02.$$

The old idea growth rate is

$$g_{A,0} = \bar{z}\bar{\ell}_0\bar{L} = 0.0005(0.02)(1000) = 0.01.$$

Since

$$1-\alpha = \frac{2}{3},$$

the old output-per-person growth rate is

$$g_{y,0} = \frac{0.01}{2/3} = 0.015.$$

Thus,

$$\boxed{g_{A,0} = 0.01, \qquad g_{y,0} = 0.015.}$$

After the policy,

$$\bar{\ell}_1 = 0.05.$$

Then

$$g_{A,1} = 0.0005(0.05)(1000) = 0.025.$$

Therefore,

$$g_{y,1} = \frac{0.025}{2/3} = 0.0375.$$

Thus,

$$\boxed{g_{A,1} = 0.025, \qquad g_{y,1} = 0.0375.}$$

Output per person is

$$y_t = A_t\left(\frac{K_t}{\bar{L}}\right)^{\alpha}(1-\bar{\ell})^{1-\alpha}.$$

Holding A_t and $K_t/\bar{L}$ fixed,

$$\frac{y_t(\bar{\ell}_1)}{y_t(\bar{\ell}_0)} = \left(\frac{1-\bar{\ell}_1}{1-\bar{\ell}_0}\right)^{1-\alpha}.$$

Substitute the numbers:

$$\frac{y_t(\bar{\ell}_1)}{y_t(\bar{\ell}_0)} = \left(\frac{0.95}{0.98}\right)^{2/3}.$$

Thus,

$$\boxed{\frac{y_t(\bar{\ell}_1)}{y_t(\bar{\ell}_0)} \approx 0.9795.}$$

So output per person falls immediately by about 2.05%.

The policy has a short-run cost because fewer workers produce final goods. But it has a long-run benefit because more workers produce ideas, raising the economy's long-run growth rate from 1.5% to 3.75%.

Marking guide: old and new g_A, 1; old and new g_y, 1; immediate output ratio, 1; interpretation, 1.